A note on fuzzy random variable generation

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Abstract. This paper presents a simple method for multi-label fuzzy random variable generation based on α-cuts. Our proposal uses fuzzy numbers and uniform random numbers to obtain a fuzzy random variable. We present and application example and discuss about different definitions of fuzzy random variables.

Keywords: Fuzzy numbers, Fuzzy random variables, Random variable generation

1 Introduction and motivation

Uncertainty coming from human-like perceptions regarding words or concepts can be handled with fuzzy sets/numbers. Its use into discrete event simulation models are a promissory field. In many simulation problems there is a lack of statistical information to use classical statistics, and one way or another such cases use information coming from experts which can be easily represented using fuzzy sets.

Our proposal focuses on an efficient method to generate fuzzy random variables in cases where multiple linguistic labels are defined. Randomness is then represented by a uniform random number and fuzzyness is represented by multiple-label fuzzy numbers. To do so, we propose to use α-levels of a fuzzy set and random numbers to obtain the realization of a fuzzy random variable.

The paper is organized as follows: Section 1 is an Introductory section. Section 2 presents some basics about fuzzy sets; in Section 3, some concepts about random variables are provided; Section 4 describes the proposed method; Section 5 presents an application example, and Section 6 presents the concluding remarks of the paper.

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2 Basics of fuzzy sets/numbers

Let $\mathcal{P}(X)$ be the class of all probabilistic sets, $\mathcal{F}(X)$ is the class of all fuzzy sets, $\mathcal{F}_1(X)$ is the class of all convex fuzzy sets, and $I = [0, 1]$ be the set of values in the unit interval. A fuzzy set namely A is characterized by a membership function $\mu_A : X \to I$ defined over a universe of discourse $x \in X$. Thus, a fuzzy set A is the set of ordered pairs $x \in X$ and its membership degree, $\mu_A(x)$, i.e.,

$$A = \{(x, \mu_A(x)) \,|\, x \in X\}. \tag{1}$$

Let us denote $\mathcal{F}_1(\mathbb{R})$ as the class of all fuzzy numbers, then a fuzzy number is defined as follows:

Definition 1 *Let $A : \mathbb{R} \to I$ be a fuzzy subset of the reals. Then $A \in \mathcal{F}_1(\mathbb{R})$ is a Fuzzy Number (FN) iff there exists a closed interval $[x_l, x_r] \neq \emptyset$ with a membership function $\mu_A(x)$ such that:*

$$\mu_A(x) = \begin{cases} c(x) & \text{for } x \in [c_l, c_r], \\ l(x) & \text{for } x \in [-\infty, x_l], \\ r(x) & \text{for } x \in [x_r, \infty], \end{cases} \tag{2}$$

where $c(x) = 1$ for $x \in [c_l, c_r]$, $l : (-\infty, x_l) \to I$ is monotonic non-decreasing, continuous from the right, i.e. $l(x) = 0$ for $x < x_l$; $l : (x_r, \infty) \to I$ is monotonic non-increasing, continuous from the left, i.e. $r(x) = 0$ for $x > x_r$.

The *α-cut* of a set $A \in \mathcal{F}_1(\mathbb{R})$, ${}^{\alpha}A$ is the set of values with a membership degree equal or greatest than α, this is:

$${}^{\alpha}A = \{x \,|\, \mu_A(x) \geqslant \alpha\} \;\forall\; x \in X, \tag{3}$$

$${}^{\alpha}A = \left[\inf_x {}^{\alpha}\mu_A(x), \sup_x {}^{\alpha}\mu_A(x)\right] = \left[\check{a}_\alpha, \hat{a}_\alpha\right], \tag{4}$$

and the *α-level* of $A \in \mathcal{F}_1(\mathbb{R})$ namely A_α are the endpoints of ${}^{\alpha}A$ i.e.

$$A_\alpha = \{x \,|\, \mu_A(x) = \alpha \text{ for some } x \in X\}, \tag{5}$$

$$A_\alpha = \left\{\inf_x {}^{\alpha}\mu_A(x), \sup_x {}^{\alpha}\mu_A(x)\right\} = \{\,\check{a}_\alpha, \hat{a}_\alpha\,\}\,. \tag{6}$$

3 Fuzzy random variables

Let us start by defining randomness in a broad sense. From a mathematical point of view, a variable X is considered *independent* if it cannot be explained by any other variable in consideration i.e. for a set of variables $X_1, \cdots, X_i, \cdots, X_m$ it is said that X_i is independent if $\nexists_{i \neq j}\{X_j | X_i = f(X_j)\}$.

Then, a variable X is *random* if it is independent with unpredictable behavior. This is often defined using probabilistic theory (see A.N. Kolmogorov [3]) as shown as follows. A probability space is a triplet $(\Omega, \mathcal{D}, \mathcal{P})$ where Ω is the sample space whose elements are $\omega \in \Omega$, $\mathcal{D}$ is a set of probable events (a.k.a σ-algebra), and $\mathcal{P} : \mathcal{D} \to I$ is a probability measure. This way, a *probabilistic random variable* $X_\omega, \omega \in \mathcal{D} \subseteq \Omega$ is characterized by $X : \Omega \to \mathcal{P}$.

Now, three main approaches to define a *fuzzy random variable (FRV)* are discussed in this paper: H. Kwarkernaak [4, 5], Puri & Ralescu [8], and Chang [1]. First we establish some additional notations that help to understand their definitions. Let T be a family of $i \in \mathbb{N}_m$ finite/countable linguistic labels/statements in which $A_i \in T$ and $\mu_{A_i} \in \mathcal{F}_1(X)$.

Kwarkernaak's definition of a FRV is based on a set $A_i \in T$ of probable fuzzy sets, as follows.

Definition 2 *A fuzzy random variable X defined on a probability space $(\Omega, \mathcal{D}, \mathcal{P})$ is characterized by a map $X : \Omega \to S$ such that*

$$\omega \overset{X}{\longmapsto} X_\omega$$

where S is the space of all piecewise continuous functions $R \to [0, 1]$.

Each element of the space S is the image of a fuzzy number i.e. $A_i(\omega) \in \mathcal{F}_1(\mathbb{R}) \in S$ where every fuzzy set A_i has an associated probability $p_i, i \in \mathbb{N}_m$, and we denote

$$i \overset{X}{\longmapsto} X^i \; \forall\, i \in \mathbb{N}_m$$

as the random variable associated to p_i for the set $A_i(\omega)$.

Puri & Ralescu [8] proposed a similar definition based on the idea of having random occurrence of a fuzzy number, as presented as follows.

Definition 3 *A fuzzy random variable $X_\alpha(\omega)$ is a function $X : \Omega \to \mathcal{F}_1(\mathbb{R}^n)$ such that*

$$\{(\omega, x) \,:\, x \in X_\alpha(\omega)\} \in \mathcal{A} \times \mathcal{B}$$

for every $\alpha \in [0, 1]$, where $X : \Omega \to \mathcal{P}(\mathbb{R}^n)$ is defined by

$$X_\alpha(\omega) = \{x \in \mathbb{R}^n \,:\, X(\omega)(x) \geqslant \alpha\}.$$

In their definition $\mathcal{A} \subseteq \mathbb{R}$ is the sample space of the probability space and $\mathcal{B}$ is the set of Borel subsets of $\mathbb{R}$ in which the membership function of $X_\alpha(\omega)$ is defined.

Now, Chang defined a fuzzy random variable (see Chang [1]) starts with defining a fuzzy topological space, as follows.

Definition 4 *A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions:*

- *Φ, $X \in T$, (null-set inclusion)*
- *If $A, B \in T$, then $A \cap B \in T$, (intersection)*
- *If $A_i \in T$ for each $i \in \mathbb{N}_m$, then $\bigcup_{\mathbb{N}_m} A_i \in T$, (union)*

where $\mathbb{N}_m$ is the index set of linguistic labels A_i that define T.

Therefore, T is called a *fuzzy topology* for X and the pair (X, T) is is called a *fuzzy topological space* or just fuzzy space.

Definition 5 *A fuzzy-random space is a triplet $(\Omega, \mathcal{D}, T)$ where Ω is the universe of discourse, $\mathcal{D}$ is a set of possible events (a.k.a σ-algebra), T is the family of fuzzy sets under consideration, and $A \in T$ is a label/concept/word, so if $(\mathcal{D}, T)$ is a topological space then it is called a fuzzy σ-algebra.*

3.1 General discussion

At a first glance all three definitions are well supported and correct, so let us go into their motivation and conceptual basis. The definition provided by H. Kwarkernaak [4, 5] defines an FRV over a probability space in the sense that a probability p_i is assigned to every set $A_i(\omega)$, so all measurability properties are based on a double map of probability and fuzzy variables. It is worth to note the following:

1. A probability space is based on probable/repeated events, so the probabilities p_i for every set $A_i(\omega)$ must be inferred or measured before.
2. The computation and integrability/measurability of every $A_i(\omega)$ are based on a convexity assumption, which is basically well defined for $\mathcal{F}_1(\mathbb{R})$.
3. The link between probabilistic uncertainty and fuzzy uncertainty is the fuzzy extension principle which is used to compute the expectation and other measures.

Puri & Ralescu [8] also define an FRV over a probability space, and also a probability p_i is assigned to a set $A_i(\omega)$ with the difference that every probable element $x^i \in \Omega$ is not considered as a real number but a fuzzy set. Again all measurability properties are based on a double map of probability and fuzzy variables. Please note the following:

1. The probabilities p_i for every element x^i must be inferred or measured before assigned to $A_i(\omega)$.
2. As in the Kwarkernaak definition, computation and integrability/measurability of every $A_i(\omega)$ are based on a convexity assumption, which is ensured for $\mathcal{F}_1(\mathbb{R})$.
3. Probabilistic uncertainty and fuzzy uncertainty are again linked by the fuzzy extension principle which is used to compute the fuzzy expectation of every probable value x^i.
4. This definition basically assigns fuzzy sets to probable events, so it can be seen as the computation of fuzzy functions of probable events.

In general, Kwarkernaak and Puri & Ralescu definitions are defined over probability spaces. This leads us to think the following: *only probable events are possible* i.e. only observed (or even observable) events shall be considered, and possible events out of the probability space should not be considered.

This implies some practical difficulties since fuzzy sets are helpful to cover unobserved and even "improbable" events. What we are intended to say is that in many cases fuzzy sets are useful to cover what experts consider the sample space should be, even if some events have never been measured/observed. For instance, life on mars is posssible but not probable yet i.e. there is no such scientific evidence to infer that life is probable out of the earth. Scientific community agree that life out of the earth is possible even if we do not have yet any objective evidence of life outside our planet.

This way, we can assert that $A_i(\omega)$ need to be observed/measured to infer all probabilities $A_i(\omega)$ which leads us to the following statement

$$\sum_i p_i(\omega) = 0 \implies A_i(\omega) \in \emptyset, \; \forall\, i \in \mathbb{N}_m$$

This statement implies that if the membership degrees $A_i(\omega)$ of every element ω are not measured/observed then they cannot be included into Ω and it implies that ω must be observed enough times to then infer $p_i(\omega)$ (possibly via experts opinions), so only probable events can be included since their membership degrees were certainly observed.

The definition provided by Puri & Ralescu is very clear on only consider fuzzy-probable events i.e. the elements of the sample space $\omega \in \Omega$ are fuzzy sets instead of singleton elements defined over a Borel set $X \in \mathcal{B}$, so $\mu_{A_\omega}(x) \in \mathcal{F}_1(\mathbb{R})$ is just an assignment to every ω. As pointed out by the authors, continuous (even non-discrete) sample spaces lead to large computations since their definition of expectation is based on the fuzzy extension principle which is well known to be NP-Hard.

Chang's definition is the simpler of the three since it considers possible events without the requirement of defining a probabilistic space to represent randomness. In Chang's approach there is the chance that a possible event can be probable if it is observed/measured while still been possible from fuzzy uncertainty theory. This can be defined as follows:

$$\sum_i A_i(\omega) = 0 \;\not\Longrightarrow\; p_i(\omega) = 0, \; \forall\, i \in \mathbb{N}_m$$

This way, randomness over Ω in Cheng's definition comes from a lack of certainty of every event, and it could be represented by any uncertainty theory. The most widely used method to represent randomness is *uniform random number generation.*

4 Fuzzy random variable generation

Fuzzy random variable generation for single fuzzy numbers e.g. single membership functions have been proposed by Varón-Gaviria et al. [9] and Pulido-López et al. [7], so a more general case is defined by Cheng's FRVs in the sense that in involves a set T of $i \in \mathbb{N}_m$ possible linguistic labels instead of only one fuzzy number.

The main idea of random variable generation is to obtain $X(\omega)$ through a uniform random number $U[0, 1]$ (further info about uniform random numbers are provided by Devroye [2], and Law & Kelton [6]). This way, we propose a simple method to obtain $X(\omega)$ using $\mu_{A_i}^{-1}$ and U to obtain $X(\omega)$.

Let $U \in I$ be a uniformly distributed random variable, $\mu_{A_i}(\omega) \in \mathcal{F}_1(\mathbb{R})\, \forall\, i \in \mathbb{N}_m$, and $\mu_{A_i}^{-1} : U \to X^i\, \forall\, i \in \mathbb{N}_m$ be invertible (or at least a Borel measurable continuous set). A realization of an FRV namely $X^i(\omega)$ regarding the i_{th} linguistic label can be obtained via its α-levels $\check{a}_{\alpha,i}$, $\hat{a}_{\alpha,i}$ which can be obtained as a function of $\alpha \in I$ as follows:

$$^{\alpha}A_i \equiv \mu_{A_i}^{-1}(\alpha) = \left[\check{a}_{\alpha,i},\, \hat{a}_{\alpha,i}\right] \forall\, i \in \mathbb{N}_m, \alpha \in I.$$

It is well known that a function of a random variable keeps the same random nature e.g. a monotonic function of a random variable U namely $g(U)$ is a random variable with the same distribution of U (see Law & Kelton [6]). This way, if we consider $U \in I$ as a uniformly distributed variable, then we have:

$$\mu_{A_i}^{-1}(\alpha) \equiv \mu_{A_i}^{-1}(U) \xrightarrow{d} U$$

where $\xrightarrow{d}$ means *convergence in distribution.*

So basically we can use a uniform random variable $U \in I$ to compute α-cuts and then consider the α-levels as realizations of $X(\omega)$. Now, as there are at most $2m$ α-levels per α-cut done over all A_i fuzzy numbers (we consider m linguistic labels and fuzzy numbers only), then we can obtain a realization $X(\omega)$ of an FRV by randomly selecting one of the possible α-levels $\check{a}_{\alpha,i}$, $\hat{a}_{\alpha,i}$ as shown in Method 1.

Procedure 1 α-cut method

Require: $m \in \mathbb{N}_+$, $\mu_{A_i} \in \mathcal{F}_1(\mathbb{R})$, $n \in \mathbb{N}_+$
  for $j : 1 \to n$ **do**
    Compute $U_{j,1} \in [0, 1]$ and $U_{j,2} \in \mathbb{N}_{2m}$
    Set $\alpha_j = U_{j,1}$
    Compute $^{\alpha}A_{i,j} = [\check{a}_{\alpha_j,i}, \hat{a}_{\alpha_j,i}]\ \forall\ i \in \mathbb{N}_m$
    Sort $\check{a}_{\alpha_j,i}, \hat{a}_{\alpha_j,i}\ \forall\ i \in \mathbb{N}_m$
    return α_j, x_j^i is the value at position $U_{j,2}$ of the sorted values $\check{a}_{\alpha_j,i}, \hat{a}_{\alpha_j,i}\ \forall\ i \in \mathbb{N}_m$
  end for

Method 1 is simple: compute the α-levels $\check{a}_{\alpha,i}$, $\hat{a}_{\alpha,i}$ of all fuzzy sets using a first uniform random number $U_1 \in I$, then sort them, generate a discrete uniform random number $U_2 \in [1, 2m]$, and finally set the realization x_j of $X^i(\omega)$ as the value of sorted $\check{a}_{\alpha,i}$, $\hat{a}_{\alpha,i}$ at the position U_2.

5 Application example

To illustrate how the proposed method works, we define an example of $m = 3$ fuzzy numbers with Gaussian membership functions $G_i(c_i, \delta_i)$ i.e. $G_1(10, 2)$, $G_2(25, 5), G_3(30, 3)$ (see Figure 1).

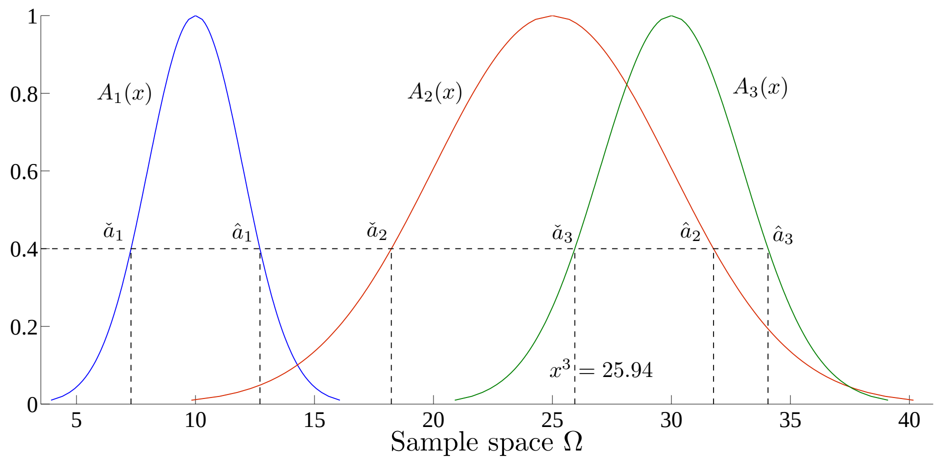

Fig. 1. Fuzzy variable with three Gaussian fuzzy numbers

Now suppose that $U_1 = 0.4$ which leads to the following values:

$$^{0.4}A_1 = [7.29, 12.71]$$
$$^{0.4}A_2 = [18.23, 31.77]$$
$$^{0.4}A_3 = [25.94, 34.06]$$

The sorted sequence is then

$$\{7.29, 12.71, 18.23, 25.94, 31.77, 34.06\}$$

Finally if we generate $U_2 = 4$ then the realization of $X^3(\omega)$ is $x^3 = 25.94$ which corresponds to $\check{a}_{\alpha,3}$ i.e. the lower α-level of the third fuzzy number with membership degree $\alpha = 0.4$. All obtained values are shown in Figure 1.

It is worth to note that we can repeat this simple procedure n times to obtain as many x^i as desired.

6 Concluding remarks

We proposed a computationally simple method for fuzzy random variable in case of multiple linguistic labels. An explanatory example was provided and some considerations about other possible approaches are provided.

The definition of FRV provided by Chang [1] is the most convenient definition since the definitions of H. Kwarkernaak [4, 5] and Puri & Ralescu [8] were based on additional probabilistic spaces.

Using two random numbers it is possible to generate a fuzzy random variable without considering additional requirements. This is specially useful in engineering applications in which fuzzy sets are used by experts to represent uncertain/unobserved data as simpler as possible.

Acknowledgment

The authors would like to thank all members of the LAMIC group of the Universidad Distrital Francisco José de Caldas in Bogotá - Colombia, the Instituto de Alta Investigación and the Department of Mathematics of the Universidad de Tarapacá in Arica - Chile.

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